Trig helps algebra.

1. Solve the system.

$$\begin{cases} x^3 - 3xy^2 = 6 \\ 3x^2y - y^3 = 8 \end{cases}.$$

Solution.

Note that $\begin{cases} x^3 - 3xy^2 = 6 \\ 3x^2y - y^3 = 8 \end{cases} \Rightarrow \dfrac{x^3 - 3xy^2}{3x^2y - y^3} = \dfrac{3}{4} \Leftrightarrow \dfrac{t^3 - 3t}{3t^2 - 1} = \dfrac{3}{4}$, where $t := \dfrac{x}{y}$.

But equation $\dfrac{t^3 - 3t}{3t^2 - 1} = \dfrac{3}{4} \Leftrightarrow 4t^3 - 12t = 9t^2 - 3 \Leftrightarrow 4t^3 - 12t - 9t^2 + 3 = 0$

have no rational solutions.

Since $\tan 3\alpha = \dfrac{\tan 2\alpha + \tan\alpha}{1 - \tan 2\alpha \tan\alpha} = \dfrac{\dfrac{2\tan\alpha}{1 - \tan^2\alpha} + \tan\alpha}{1 - \dfrac{2\tan\alpha}{1 - \tan^2\alpha} \cdot \tan\alpha} = \dfrac{\tan^3\alpha - 3\tan\alpha}{3\tan^2\alpha - 1}$

then denoting $\theta := \tan^{-1}(t)$ we obtain $\dfrac{t^3 - 3t}{3t^2 - 1} = \dfrac{3}{4} \Leftrightarrow \dfrac{\tan^3\theta - 3\tan\theta}{3\tan^2\theta - 1} = \dfrac{3}{4} \Leftrightarrow$

$\tan 3\theta = \dfrac{3}{4} \Leftrightarrow \theta = \dfrac{1}{3}\tan^{-1}\left(\dfrac{3}{4}\right)$. Then $t = \tan\left(\dfrac{1}{3}\tan^{-1}\left(\dfrac{3}{4}\right)\right)$ and

$3x^2y - y^3 = 8 \Leftrightarrow y^3(3t^2 - 1) = 8 \Leftrightarrow y = \dfrac{2}{\sqrt[3]{3t^2 - 1}} \Rightarrow x = \dfrac{2t}{\sqrt[3]{3t^2 - 1}}$.